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The evolution of the Weyl and Maxwell fields in curved space-times

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Abstract. The covariant Weyl (spin $s = \frac{1}{2}$) and Maxwell ($s = 1$) equations in certain local charts $(\mathcal{U}, \tilde{\varphi})$ of a space-time $(\mathcal{M}, \boldsymbol{g})$ are considered. It is shown that the condition $g^{00}(x) > 0 \ \forall x \in \mathcal{U}$ is necessary and sufficient to rewrite them in a unified manner as evolution equations $\partial_t \phi = L_{(s)} \phi$. Here $L_{(s)}$ is a linear first order differential operator on the pre-Hilbert space $\left(C_0^\infty(U_t, \mathbb{C}^{2s+1}), \langle \cdot, \cdot \rangle\right)$, where $U_t \subset \mathbb{R}^3$ is the image of the coordinate map of the spacelike hypersurface t = const, and $\langle \phi, \psi \rangle = \int_{U_t} \phi^\star Q \psi \, \mathrm{d}^{(3)}\boldsymbol{x}$ with a suitable Hermitian $n \times n$-matrix $Q = Q(t, \boldsymbol{x})$. The total energy of the spinor field ϕ with respect to U_t is then simply given by $E = \langle \phi, \phi \rangle$. In this way inequalities for the energy change rate with respect to time, $\partial_t \|\phi\|^2 = 2 \operatorname{Re} \langle \phi, L_{(s)} \phi \rangle$, are obtained. As an application, the Kerr-Newman black hole is studied, yielding quantitative estimates for the energy change rate. These estimates especially confirm the energy conservation of the Weyl field and the well-known *superradiance* of electromagnetic waves.

1. Introduction

Energy in general relativity is based on a local concept. It is a weaker notion than one is used to in special relativity, where total energy and total momentum are considered, in essence, as integrals of energy-momentum tensor densities over the hypersurfaces t = const. These integrals yield a 4-vector that transforms by the Lorentz group under changes of reference frames in Minkowski space-time.

Here we generalize this concept. We suppose a space-time $(\mathcal{M}, \boldsymbol{g})$ with a foliation $\{\mathcal{L}_{t \in I}\}$, $I \subset \mathbb{R}$, where each leaf $\mathcal{L}_t$ is a spacelike hypersurface. Just as in special relativity, we regard the integral $E = \int_{\mathcal{L}_t} T_{ij} N^i N^j \mathrm{d}^{(3)}\boldsymbol{x}$ as the total energy with respect

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to $\mathcal{L}_t$ at time $t \in I$, where T_{ij} is the energy-momentum tensor of the considered matter and N^j the unit normal to $\mathcal{L}_t$.

For quantitative calculations it is necessary to use coordinates. Thus we restrict ourselves to a suitable open subset $\mathcal{U} \subset \mathcal{M}$ with coordinates $(x^0, ..., x^3) : \mathcal{U} \to U \subset \mathbb{R}^4$, $x^0 = ct$, c the speed of light, where the hypersurfaces $\{t = \text{const}\}$ are $\mathcal{L}_t \cap \mathcal{U}$. One important result is the fact that a necessary and sufficient criterion for the hypersurfaces $t = \text{const}$ to be spacelike is the positiveness of the metric tensor component g^{00},

$$g^{00}(x) > 0 \qquad \text{for each } x \in \mathcal{U}. \tag{1.1}$$

This condition is obviously true in Minkowski space-time, but surprisingly it is also valid in the whole outer space of a rotating black hole in Boyer-Lindquist coordinates, especially in the ergosphere (where $g_{00} < 0\,!$). In this way, using a chart compatible with the foliation and obeying the condition $g^{00} > 0$, we rewrite the covariant Weyl and Maxwell equations on $\mathcal{M}$ as evolution equations. These equations allow us to calculate bounds for the rate of change of the energy E during the progress of time t, depending only on the components of the respective differential equations. Whereas for the flat case one expects energy conservation for test fields, i.e. an isometry of the evolution for the energy norm (i.e. $\partial E/\partial t = 0$), the interactions with a general gravitational field may result in dissipation, or amplification, of energy. These effects will be studied quantitatively.

This article is organized as follows: After presenting an important elementary property of Newman-Penrose tetrads in section 2, we give some essential implications of condition (1.1), $g^{00} > 0$, in section 3. Section 4 collects mathematical properties of general linear differential operators with matrix-valued coefficients, whereas section 5 provides a physical discussion of the definition of the total energy E of a wave field with respect to the hypersurface $U_t = \{(x^1, x^2, x^3)\}$. Essentially, the contravariant component T^{00}, the projection of the energy-momentum tensor T_{ij} on the vector field $X^j = g^{0j}$, $T^{00} = T_{ij}X^iX^j$, is the energy density with respect to the hypersurface U_t, because X^j is everywhere orthogonal to U_t by lemma 3.1 below. Besides, T^{00} is in general the only component of T_{ij} that is always positive on U_t; for example, $T^0_0 = T^0_j(\partial_t)^j$ or T_{00} for the Maxwell field in the ergosphere of the Kerr-Newman space-time are negative, as is shown in section 7. The unit normal to U_t being given by $N^j = X^j/\sqrt{g^{00}}$, we define $E = \int_{U_t} T^{00}/g^{00} \mathrm{d}^{(3)}\boldsymbol{x}$ as the total energy of ϕ with respect to the hypersurface U_t. Of course, this quantity is hypersurface-dependent. But paying this price, we gain the possibility of rewriting the Weyl and Maxwell equations as evolution equations, $\partial_t \phi = L_{(s)}\phi$, where $s = \frac{1}{2}$ for the Weyl field, and $s = 1$ for the Maxwell field, as is done in section 6. The originally covariant equations, once transported into $\mathbb{R}^4$ by the coordinate map, can now be treated as usual linear differential equations in $U \subset \mathbb{R}_{(t)} \times \mathbb{R}^3_{(\boldsymbol{x})}$. Here it is standard to calculate changes of the energy E with respect to time, $\partial_t E = 2 \operatorname{Re} \langle \phi, L_{(s)}\phi \rangle$, as long as it *is* a norm square, i.e. $E \geq 0$.

We emphasize that the energy change rates are independent of the special definition of the total energy E (if only it is positive), because they depend only on the coefficients of the differential equations. This may be recognized noticing the central inequalities of this article, (6.10) and (6.21).

Finally, the restriction $g^{00} > 0$ is general enough to include such physically important cases like the outer space of a rotating charged black hole, as considered in section 7, or the Robertson-Walker universes, cf. de Vries (1994), for which the corresponding charts are even global (up to a set of measure zero).

To summarize, the criterion $g^{00} > 0$ does not only enable us to gain evolution equations for the Weyl and Maxwell fields, but also guarantees that the total energy with respect to U_t is positive. Hence there is a natural construction of a pre-Hilbert space $\big(C_0^\infty(U_t, \mathbb{C}^{2s}), \langle\cdot,\cdot\rangle\big)$ of the spinor functions. In this way standard functional analytic methods can be applied to general relativity, yielding new quantitative results in a rather simple manner.

2. Newman-Penrose tetrads

In the sequel we consider space-times $(\mathcal{M}, \boldsymbol{g})$ with a Lorentz metric of signature -2, e.g. Hawking and Ellis (1973). Moreover we use the convention to sum over indices occuring twice, counting Latin indices from 0 to 3 and Greek ones from 1 to 3, cf. Landau and Lifschitz (1975).

Let be $(\mathcal{M}, \boldsymbol{g})$ a general space-time, and $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ a *Newman-Penrose tetrad*, i.e. a collection of two real future directed null vector fields, $\boldsymbol{l}$ and $\boldsymbol{n}$, and a complex null vector field $\boldsymbol{m}$ with its complex conjugate $\bar{\boldsymbol{m}}$, satisfying the orthonormality relations

$$l_j l^j = n_j n^j = m_j m^j = \bar{m}_j \bar{m}^j = 0,$$

$$l_j n^j = -m_j \bar{m}^j = 1,$$

$$l_j m^j = l_j \bar{m}^j = n_j m^j = n_j \bar{m}^j = 0.$$

We mention the relation

$$g^{ij} = l^i n^j + n^i l^j - m^i \bar{m}^j - \bar{m}^i m^j, \tag{2.1}$$

cf. Newman and Penrose (1962). Let $e_{(a)}{}^j$ and $e^{(a)j}$ for $a = 1, 2, 3, 4$ be given by

$$\begin{aligned} e_{(1)}{}^j = e^{(2)j} = l^j, \quad e_{(2)}{}^j = e^{(1)j} = n^j, \\ e_{(3)}{}^j = -e^{(4)j} = m^j, \quad e_{(4)}{}^j = -e^{(3)j} = \bar{m}^j. \end{aligned} \tag{2.2}$$

Relating the Ricci rotation coefficients of the Newman-Penrose tetrad,

$$\gamma_{abc} = e_{(a)}{}^i e_{(b)i;\,j} e_{(c)}{}^j,$$

to the spin coefficients we get the equations

$$\begin{aligned} &\kappa = \gamma_{311}, && \varrho = \gamma_{314}, && \varepsilon = \tfrac{1}{2}(\gamma_{211} + \gamma_{341}), \\ &\sigma = \gamma_{313}, && \mu = \gamma_{243}, && \gamma = \tfrac{1}{2}(\gamma_{212} + \gamma_{342}), \\ &\lambda = \gamma_{244}, && \tau = \gamma_{312}, && \alpha = \tfrac{1}{2}(\gamma_{214} + \gamma_{344}), \\ &\nu = \gamma_{242}, && \pi = \gamma_{241}, && \beta = \tfrac{1}{2}(\gamma_{213} + \gamma_{343}), \end{aligned} \tag{2.3}$$

e.g. Chandrasekhar (1983) or Newman and Penrose (1962).

Lemma 2.1. *For a Newman-Penrose tetrad it follows*

$$\nabla_j l^j = 2\,\mathrm{Re}(\varepsilon - \varrho), \quad \nabla_j n^j = 2\,\mathrm{Re}(\mu - \gamma), \quad \nabla_j m^j = \beta - \tau + \bar{\pi} - \bar{\alpha}. \tag{2.4}$$

P r o o f. By $\gamma_{1a2} + \gamma_{2a1} - \gamma_{3a4} - \gamma_{4a3} = (l^i n^j + n^i l^j - m^i \bar{m}^j - \bar{m}^i m^j)\nabla_j e_{(a)i}$ and by (2.1) we see

$$\nabla_j e_{(a)}{}^j = \gamma_{1a2} + \gamma_{2a1} - \gamma_{3a4} - \gamma_{4a3}, \tag{2.5}$$

From the antisymmetry of the Ricci rotation coefficients in the first two indices, $\gamma_{abc} = -\gamma_{bac}$, (e.g. Chandrasekhar 1983, or Landau & Lifschitz 1975) it follows $\gamma_{aab} = 0$, and hence with $\varepsilon + \bar{\varepsilon} = \gamma_{211}$,

$$\varepsilon + \bar{\varepsilon} - \varrho - \bar{\varrho} = \gamma_{211} - \gamma_{314} - \gamma_{413} = \gamma_{111} + \gamma_{211} - \gamma_{314} - \gamma_{413}.$$

Setting $a = 1$ in (2.5) we receive the first equation in (2.4). Similarly we get $\gamma + \bar{\gamma} = \gamma_{212}$ and $\beta - \bar{\alpha} = \gamma_{343}$, i.e. with $a = 2$ and $a = 3$ the last two equations in (2.4). □

3. The condition $g^{00} > 0$

We consider some properties of the components of the metric tensor in a general local coordinate system of an arbitrary space-time. In the sequel we denote $\partial_j = \partial/\partial x^j$ for $j = 0, 1, 2, 3$.

Due to the fundamental relation $g_{ij}g^{jk} = \delta_i{}^k$ we have

$$g_{\alpha\beta}g^{\beta\gamma} + g_{0\alpha}g^{0\gamma} = \delta_\alpha{}^\gamma, \tag{3.1}$$

$$g_{\alpha\beta}g^{0\beta} + g_{0\alpha}g^{00} = 0, \tag{3.2}$$

$$g_{0\beta}g^{0\beta} + g_{00}g^{00} = 1 \tag{3.3}$$

$(\alpha, \beta, \gamma = 1, 2, 3)$. Especially

$$g_{\alpha i}g^{0i} = 0. \tag{3.4}$$

This means geometrically that in every space-time, for each $\alpha = 1, 2, 3$, the two four-vectors $\boldsymbol{X}$ and ∂_α, with the components $X^i = g^{0i}$ and $\partial_\alpha{}^i = \delta_\alpha{}^i$, are orthogonal to each other with respect to the metric $\boldsymbol{g}$. By (3.2) we have $-g_{\alpha\beta}g^{0\beta} = g_{0\alpha}g^{00}$, thus summed over α

$$-g_{\alpha\beta}g^{0\alpha}g^{0\beta} = g_{0\alpha}g^{0\alpha}g^{00}. \tag{3.5}$$

Lemma 3.1. *For each $x \in \mathcal{M}$ we have the following assertions:*
(a) *The 3×3-matrix $\big(g_{\alpha\beta}(x)\big)$ is negative definite if and only if $g^{00}(x) > 0$.*
(b) *If $g^{00}(x) > 0$, then either $g_{0\alpha}(x) = g^{0\alpha}(x) = 0$, or $g_{0\alpha}(x)g^{0\alpha}(x) > 0$.*

P r o o f. (a) $(g_{\alpha\beta})$ is negative definite $\iff$ $\mathrm{span}(\partial_1, \partial_2, \partial_3)$ is spacelike

$$\underset{(3.4)}{\iff} \boldsymbol{X} \text{ given by } X^i = g^{0i} \text{ is timelike} \iff \boldsymbol{g}(\boldsymbol{X},\boldsymbol{X}) = g_{ij}g^{0i}g^{0j} = g^{00} > 0.$$

(b) By (a) $g_{\alpha\beta}$ is negative definite. If $g_{0\alpha}g^{0\alpha} \neq 0$, then $g_{0\alpha}g^{0\alpha} > 0$ by (3.5); if else $g_{0\alpha}g^{0\alpha} = 0$, then $g^{0\alpha} = 0$ by (3.5), and hence by (3.2) $g_{0\alpha} = 0$. □

Lemma 3.2. *Let $\mathcal{U}$ be connected and $(\mathcal{U}, \tilde{\varphi})$ be a chart, such that $g^{00} > 0$ in $\mathcal{U}$, and let $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ be a Newman-Penrose tetrad. Furthermore let the vector field ∂_0 be timelike and future directed in a special point $x_\star \in \mathcal{U}$. Then in each point of $\mathcal{U}$*

$$l^0, n^0, l_0, n_0 > 0. \tag{3.6}$$

P r o o f. Because ∂_0 is timelike in $x_\star$, we have $g_{00}(x_\star) > 0$. By (2.1) we know $g^{00} = 2(l^0 n^0 - m^0 \bar{m}^0)$, and hence $l^0(x_\star)n^0(x_\star) > 0$. But being continuous on the connected set $\mathcal{U}$, both l^0 and n^0 have the same sign and do not change it anywhere. Now, the vector field $X^j = g^{0j}$ is timelike by (3.4) and lemma 3.1, and future directed in $x_\star$ by $\boldsymbol{g}_{x_\star}(\partial_0, \boldsymbol{X}) = g_{00}g^{00} > 0$. By definition, $\boldsymbol{l}$ and $\boldsymbol{n}$ are future directed, hence $\boldsymbol{g}_{x_\star}(\boldsymbol{X}, \boldsymbol{l}) = g^{0j}(x_\star)l_j(x_\star) = l^0(x_\star) > 0$, and analogously $n^0(x_\star) > 0$. l^0 and n^0 do not change the sign on $\mathcal{U}$, and thus we have in general $l^0(x), n^0(x) > 0$ $\forall x \in \mathcal{U}$. Because $\boldsymbol{l}$ and $\boldsymbol{n}$ are lightlike, we conclude by the negative definiteness $l_0 l^0 = -l_\alpha l^\alpha = -g_{\alpha\beta}l^\alpha l^\beta > 0$ and $n_0 n^0 = -g_{\alpha\beta}n^\alpha n^\beta > 0$. □

4. Linear operators in space-times

For a general space-time $(\mathcal{M}, \boldsymbol{g})$ we consider a timelike future directed vector field $\boldsymbol{X}$ on $\mathcal{M}$, $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{X}) > 0$. Let $\mathcal{U} \subset \mathcal{M}$ be an open connected subset such that there exists an onto C^∞-function $f : \mathcal{U} \to I \subset \mathbb{R}$ whose gradient is the vector field $\boldsymbol{X}$ divided by c, i.e. $c\,\mathrm{d}f = \boldsymbol{g}(\boldsymbol{X}, \cdot)$, (Choquet-Bruhat *et al.* 1982, pp. 285). By $c\,\mathrm{d}f(\boldsymbol{X}) = \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{X}) > 0$, f has no critical points. Thus for $t \in I$ the level surfaces $\mathcal{L}_t = \{x \in \mathcal{U} \mid f(x) = t\}$ are hypersurfaces of $\mathcal{U}$. They are spacelike, because the gradient of f is timelike in each point $x \in \mathcal{U}$. $\bigcup_{t \in I} \mathcal{L}_t \cap \mathcal{U}$ is a foliation of $\mathcal{U}$, Tondeur (1988). Then we call $f(x)$ the *time* in $x \in \mathcal{U}$. We mention that f can be extended to a global function $f : \mathcal{M} \to \mathbb{R}$, iff the stable causality condition holds on the space-time (Hawking & Ellis 1973, pp. 198). For each $t \in I$ we assume $\mathcal{L}_t \cap \mathcal{U}$ connected, and globally parametrized by $\tilde{\varphi}(x) = (ct, \boldsymbol{x}) = (ct, x^1, x^2, x^3)$. This means that $(\mathcal{U}, \tilde{\varphi})$ is a local coordinate system of $\mathcal{M}$. Let now $U := \tilde{\varphi}(\mathcal{U})$ be the image of the coordinate map, and

$$U_t := \{\boldsymbol{x} \in \mathbb{R}^3 \mid (ct, \boldsymbol{x}) \in \tilde{\varphi}(\mathcal{U})\} \qquad \forall t \in I. \tag{4.1}$$

We have $U_t \cong \mathcal{L}_t \cap \mathcal{U}$, as well as $\bigcup_{t \in I}\{ct\} \times U_t \cong \tilde{\varphi}(\mathcal{U})$. By lemma 3.1 (a) we see that U_t is spacelike, if and only if (1.1) is valid.

For $n \in \mathbb{N}$ we consider the pre-Hilbert space $\big(C_0^\infty(U_t, \mathbb{C}^n), \langle\cdot,\cdot\rangle\big)$ of the smooth functions $\phi : U_t \to \mathbb{C}^n$ with compact support and the scalar product $\langle\cdot,\cdot\rangle : C_0^\infty(U_t, \mathbb{C}^n) \times C_0^\infty(U_t, \mathbb{C}^n) \to \mathbb{C}$,

$$\langle \phi, \psi \rangle = \int_{U_t} \phi^\star Q \psi \sqrt{|g|}\, \mathrm{d}\boldsymbol{x}. \tag{4.2}$$

Here $g = \det g_{ij}$, and $Q = Q(t, \boldsymbol{x})$ is a Hermitian positive definite $n \times n$-matrix with C^0-entries, denoted $Q = (q_{ij})_{i,j=1,\dots,n}$. Let furthermore $\mathcal{H} = \mathcal{H}_t = L^2(U_t)^n$ be the completion of $C_0^\infty(U_t, \mathbb{C}^n)$ with respect to the norm $\|\cdot\|$ induced by the scalar product $\langle\cdot,\cdot\rangle$, cf. CHERNOFF (1973). We abbreviate for notational convenience

$$\mathrm{d}^{(3)}\boldsymbol{x} = \sqrt{|g|}\,\mathrm{d}\boldsymbol{x}. \tag{4.3}$$

We denote by I_n the $n \times n$ identity matrix, and $\langle\phi,\psi\rangle_{I_n} = \int_{U_t} \phi^\star\psi\,\mathrm{d}^{(3)}\boldsymbol{x}$. Notice $\langle\phi,\psi\rangle = \langle\phi, Q\psi\rangle_{I_n}$.

Definition 4.1. For functions $\phi,\psi\colon U \to \mathbb{C}^n$ we define

$$\{\!\!\{\phi,\psi\}\!\!\}^{\pm}(t,\boldsymbol{x}) := \sum_{i,j=1}^{n} (\bar\phi_i q_{ij} \psi_j)^{\pm}(t,\boldsymbol{x}), \tag{4.4}$$

where $f^+(t,\boldsymbol{x}) = \max\bigl(0, \operatorname{Re} f(t,\boldsymbol{x})\bigr)$ and $f^-(t,\boldsymbol{x}) = \min\bigl(0, \operatorname{Re} f(t,\boldsymbol{x})\bigr)$ for a function $f\colon U \to \mathbb{C}$. It follows $\operatorname{Re}\langle\phi,\psi\rangle = \int (\{\!\!\{\phi,\psi\}\!\!\}^+ + \{\!\!\{\phi,\psi\}\!\!\}^-)\,\mathrm{d}^{(3)}\boldsymbol{x}$.

Lemma 4.2. *Consider $t \in I$ as given. Let $H = H(t,\boldsymbol{x})$ be a diagonalizable complex $n \times n$-matrix with real eigenvalues $\lambda_1,\dots,\lambda_n \in C^0(U,\mathbb{R})$, and denote $\lambda_{\min}(t,\boldsymbol{x}) = \min \lambda_i(t,\boldsymbol{x})$ and $\lambda_{\max}(t,\boldsymbol{x}) = \max\lambda_i(t,\boldsymbol{x})$ in each point $(t,\boldsymbol{x}) \in U$. For $\phi,\psi \in \mathcal{H}$ we then have the pointwise inequalities*

$$\begin{aligned}\lambda_{\min}\{\!\!\{\phi,\psi\}\!\!\}^+ + \lambda_{\max}\{\!\!\{\phi,\psi\}\!\!\}^- &\le \operatorname{Re}(\phi^\star QH\psi)\\ &\le \lambda_{\min}\{\!\!\{\phi,\psi\}\!\!\}^- + \lambda_{\max}\{\!\!\{\phi,\psi\}\!\!\}^+.\end{aligned} \tag{4.5}$$

If $\{\!\!\{\phi,\psi\}\!\!\}^{\pm} = \{\!\!\{\phi',\psi'\}\!\!\}^{\pm}$ for $\phi,\psi,\phi',\psi' \in \mathcal{H}$, then

$$\{\!\!\{\phi, H\psi\}\!\!\}^{\pm} = \{\!\!\{\phi', H\psi'\}\!\!\}^{\pm}. \tag{4.6}$$

P r o o f. Let S be a unitary matrix such that $D := SHS^\star$ is diagonal. Defining $\eta := Q^{-1}SQ\phi$, $\chi := S\psi$ we have $\eta^\star Q\chi = \phi^\star Q\psi$ and $\eta^\star QD\chi = \phi^\star QH\psi$. Because D is diagonal with entries λ_j, we see with $Q = (q_{ij})_{i,j}$ that

$$\operatorname{Re}(\eta^\star QD\chi) = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \operatorname{Re}(\bar\eta_i q_{ij}\chi_j).$$

Hence $\operatorname{Re}(\eta^\star QD\chi) = \sum_j \lambda_j \sum_i (\bar\eta_i q_{ij}\chi_j)^+ + \sum_j \lambda_j \sum_i (\bar\eta_i q_{ij}\chi_j)^-$. By

$$\begin{aligned}\lambda_{\min}\{\!\!\{\eta,\chi\}\!\!\}^+ &\le \textstyle\sum\lambda_j\sum(\bar\eta_i q_{ij}\chi_j)^+ \le \lambda_{\max}\{\!\!\{\eta,\chi\}\!\!\}^+,\\ \lambda_{\max}\{\!\!\{\eta,\chi\}\!\!\}^- &\le \textstyle\sum\lambda_j\sum(\bar\eta_i q_{ij}\chi_j)^- \le \lambda_{\min}\{\!\!\{\eta,\chi\}\!\!\}^-,\end{aligned}$$

(4.5) is proved. Suppose now $\{\!\!\{\phi,\psi\}\!\!\}^{\pm} = \{\!\!\{\phi',\psi'\}\!\!\}^{\pm}$, and let η' and χ' be given by ϕ' and ψ' analogously as η and χ above. Then we have $\{\!\!\{\phi,\psi\}\!\!\}^{\pm} - \{\!\!\{\phi',\psi'\}\!\!\}^{\pm} = \sum(\bar\eta_i q_{ij}\chi_j)^{\pm} - (\bar\eta'_i q_{ij}\chi'_j)^{\pm} = 0$, thus $(\bar\eta_i q_{ij}\chi_j)^{\pm} - (\bar\eta'_i q_{ij}\chi'_j)^{\pm} = 0$ (no summation!), i.e. $\{\!\!\{\phi,H\psi\}\!\!\}^{\pm} - \{\!\!\{\phi',H\psi'\}\!\!\}^{\pm} = \sum\lambda_j\bigl((\bar\eta_i q_{ij}\chi_j)^{\pm} - (\bar\eta'_i q_{ij}\chi'_j)^{\pm}\bigr) = 0$. □

As a simple consequence of (4.5) for $\psi = \phi$ we have, by $\{\!\!\{\phi,\phi\}\!\!\}^+ = \phi^\star Q\phi > 0$,

$$\lambda_{\min}\,\phi^\star Q\phi \le \operatorname{Re}(\phi^\star QH\phi) \le \lambda_{\max}\,\phi^\star Q\phi. \tag{4.7}$$

Lemma 4.3. *For $\nu \in \{1,2,3\}$ let $M^\nu = M^\nu(t,\boldsymbol{x})$ be three Hermitian $n \times n$-matrices, and let $B : C_0^\infty(U_t, \mathbb{C}^n) \to C_0^\infty(U_t, \mathbb{C}^n)$ be the differential operator $B = -M^\nu \partial_\nu$. Then for $\phi \in C_0^\infty(U_t, \mathbb{C}^n)$ we have (cf. (4.3))*

$$2\,\mathrm{Re}\,\langle \phi, B\phi\rangle_{I_n} = \int_{U_t} \phi^\star \frac{1}{\sqrt{|g|}}\,\partial_\nu\big(\sqrt{|g|}\,M^\nu\big)\phi\,\mathrm{d}^{(3)}\boldsymbol{x}.$$

P r o o f. We define the vector field $F_\phi : U_t \to \mathbb{R}^3$ with the components $F_\phi^\nu = \phi^\star \cdot M^\nu \cdot \phi$. The three matrices M^ν being Hermitian, F_ϕ^ν is real. We consider $\mathrm{d}^{(3)}\boldsymbol{x}$ as the 3-form $\sqrt{|g|}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ on the open subset $U_t \subset \mathbb{R}^3$. For the dual $*F_\phi$ we then have $\mathrm{d} * F_\phi = \partial_\nu(\sqrt{|g|}\;F_\phi^\nu)\;\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, e.g. Choquet-Bruhat *et al.* (1982). By Stokes' theorem we have $\int_{U_t} \mathrm{d} * F_\phi = \int_{\partial U_t} *F_\phi$. But the boundary integral vanishes ($\mathrm{supp}\,\phi \subset\subset U_t$), and hence with

$$\partial_\nu(\sqrt{|g|}\,F_\phi^\nu) = \sqrt{|g|}\,(\partial_\nu\phi^\star M^\nu\phi + \phi^\star M^\nu \partial_\nu\phi) + \phi^\star\partial_\nu(\sqrt{|g|}\,M^\nu)\phi$$

it follows

$$\int_{U_t}\big(\partial_\nu\phi^\star M^\nu\phi + \phi^\star M^\nu\partial_\nu\phi\big)\mathrm{d}^{(3)}\boldsymbol{x} = -\int_{U_t}\phi^\star\frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}\,M^\nu)\phi\,\mathrm{d}^{(3)}\boldsymbol{x}.$$

$2\,\mathrm{Re}\,\langle\phi, B\phi\rangle_{I_n} = -\int(\partial_\nu\phi^\star M^\nu\phi + \phi^\star M^\nu\partial_\nu\phi)\mathrm{d}^{(3)}\boldsymbol{x}$ completes the proof. □

Lemma 4.4. *Let $\mathcal{H}$ be a complex Hilbert space, and let A_1 and A_2 be two linear operators with domains $D(A_1), D(A_2) \subset \mathcal{H}$ such that $\langle\phi, A_1\phi\rangle = \langle\phi, A_2\phi\rangle$ for each $\phi \in D(A_1)\cap D(A_2)$. Then $A_1\phi = A_2\phi$ for each $\phi \in D(A_1)\cap D(A_2)$. If moreover $D(A_1)\cap D(A_2)$ is dense in $\mathcal{H}$, then $A_1 = A_2$.*

P r o o f. Define $A := A_1 - A_2$ with domain $D(A) = D(A_1)\cap D(A_2)$. Then for $\phi \in D(A)$ we have $\langle\phi, A_1\phi\rangle = \langle\phi, A_2\phi\rangle \Leftrightarrow \langle\phi, A\phi\rangle = 0$. $q : D(A) \to \mathbb{R}$ defined by $q(\phi) = \langle\phi, A\phi\rangle$ is a quadratic form on $D(A)$ vanishing identically, $q(\phi) \equiv 0$. The sesquilinear form $s : D(A)\times D(A) \to \mathbb{C}$, $s(\psi,\phi) = \langle\psi, A\phi\rangle$ also vanishes identically by the polarisation identity, because $D(A)$ is a complex vector space, e.g. Weidmann (1976). Hence $\langle\psi, A\phi\rangle \equiv 0\ \forall\phi,\psi \in D(A)$. Thus $A\phi = 0$ for $\phi \in D(A)$. If $D(A)$ is dense in $\mathcal{H}$, then $A = 0$. □

Remark. In the proof it is essential that $D(A_1)\cap D(A_2)$ is a *complex* vector space, because in real spaces there do exist linear operators A ("rotations by $\pm\frac{\pi}{2}$") such that the associated quadratic form vanishes identically, $q(\phi) = \langle\phi, A\phi\rangle = 0$, but the sesquilinear form $s(\phi,\psi) = \langle\phi, A\psi\rangle$ does not.

Theorem 4.5. *Let $(\mathcal{U}, \tilde\varphi)$ be a local chart of a space-time $(\mathcal{M}, \boldsymbol{g})$ satisfying* (1.1), *and let be U and U_t be given as above, eq.* (4.1). *Moreover we consider the matrix-valued C^1-maps $M^j, Z : U \to M(n\times n, \mathbb{C})$, $j = 0,1,2,3$, and assume for each $(t,\boldsymbol{x}) \in U$ the following:*

(i) $M^0(t,\boldsymbol{x})$ *is Hermitian with positive eigenvalues, and* $\lambda_{\max}(t,\boldsymbol{x})$ *is the maximal eigenvalue of the inverse* $(M^0)^{-1}$.

(ii) *The three matrices* $(M^\nu)(t,\boldsymbol{x}),\ \nu=1,2,3,$ *are Hermitian.*

(iii) $\mu_{\min}$ *and* $\mu_{\max}$ *are the minimal and maximal eigenvalue, respectively, of the* $n\times n$*-matrix*

$$H = H(t,\boldsymbol{x}) := \frac{1}{\sqrt{|g|}}\,\partial_\nu\big(\sqrt{|g|}\,M^\nu\big) - Z - Z^\star. \tag{4.8}$$

Then, for the first order differential operator $L\colon C_0^\infty(U_t,\mathbb{C}^n)\to C_0^\infty(U_t,\mathbb{C}^n)$,

$$L = -\big(M^0(t,\boldsymbol{x})\big)^{-1}\big(M^\nu(t,\boldsymbol{x})\partial_\nu + Z(t,\boldsymbol{x})\big), \tag{4.9}$$

and for $\phi\in C_0^\infty(U_t,\mathbb{C}^n)$ *we have*

$$2\,\mathrm{Re}\,\langle\phi,L\phi\rangle = \mathrm{Re}\langle\phi,(M^0)^{-1}H\phi\rangle, \tag{4.10}$$

and the inequalities

$$\langle\phi,\lambda_{\max}\mu^-_{\min}\phi\rangle \le 2\,\mathrm{Re}\,\langle\phi,L\phi\rangle \le \langle\phi,\lambda_{\max}\mu^+_{\max}\phi\rangle. \tag{4.11}$$

P r o o f. Let $B=-M^\nu\partial_\nu$. Then $(B+\widetilde{B}-Z-Z^\star)\phi=H\phi$ for $\phi\in C_0^\infty(U_t,\mathbb{C}^n)$ by lemma 4.3 and 4.4, where $\widetilde{B}$ is the formal adjoint of B with respect to $\langle\cdot,\cdot\rangle_{I_n}$. Especially, $2\,\{\phi,(B-Z)\phi\}^\pm=\{\phi,H\phi\}^\pm$, hence $2\,\{\phi,(M^0)^{-1}(B-Z)\phi\}^\pm=\{\phi,(M^0)^{-1}H\phi\}^\pm$ by (4.6), i.e. $2\,\{\phi,L\phi\}^\pm=\{\phi,(M^0)^{-1}H\phi\}^\pm$, pointwise. Integration yields (4.10). With (4.7) we see immediately $\mu_{\min}\,\phi^\star Q\phi\le\mathrm{Re}\,(\phi^\star QH\phi)\le\mu_{\max}\,\phi^\star Q\phi$. This gives, by (4.5) and (4.6), and denoting the minimal eigenvalue of $(M^0)^{-1}$ by $\lambda_{\min}$,

$$\begin{aligned}(\lambda_{\min}\,\mu^+_{\min}+\lambda_{\max}\,\mu^-_{\min})\ \phi^\star Q\phi &\le \mathrm{Re}\,\big(\phi^\star Q(M^0)^{-1}H\phi\big)\\ &\le (\lambda_{\min}\,\mu^-_{\max}+\lambda_{\max}\,\mu^+_{\max})\ \phi^\star Q\phi.\end{aligned}$$

Because $\lambda_{\min}$ and $\lambda_{\max}$ are positive and $\mu^-_{\dots}\le 0\le\mu^+_{\dots}$, (4.11) is proved. □

5. The energy with respect to U_t

In general relativity, the energy properties of matter are represented by an energy-momentum tensor T_{ij}. Suppose a spacelike hypersurface $\Sigma\subset\mathcal{M}$, with the unit normal N^j. Then $J_i=T_{ij}N^j$ is the covariant energy-momentum vector with respect to Σ. This is a 1-form, $\boldsymbol{J}=J_i\,\mathrm{d}x^i$. Its dual, a 3-form, is given by

$$\star\boldsymbol{J}=\sqrt{|g|}\,\sum_{j=0}^{3}(-1)^j J^j\mathrm{d}x^0\wedge\ldots\wedge\widehat{\mathrm{d}x^j}\wedge\ldots\wedge\mathrm{d}x^3$$

(Choquet-Bruhat *et al.*, 1982, pp. 316). If the coordinates $\{x^0, ..., x^3\}$ are chosen such that $\Sigma = \Sigma(t) = \{x \in \mathcal{M} \mid x^0 = ct\}$ for $t \in \mathbb{R}$, then

$$\star \boldsymbol{J} = J^0 \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = T^0_j N^j \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

on Σ, because $\mathrm{d}x^0 = 0$. Hence we call $E = \int_\Sigma \star \boldsymbol{J}$ the *total energy* with respect to the hypersurface Σ. In general, E is not independent of the hypersurface Σ. However, if the N^j is a Killing field, and if only Cauchy surfaces are considered, E is hypersurface-independent and constant on $\mathcal{M}$(Wald 1984, pp. 284, or Hawking and Ellis 1973, pp. 206).

Let U_t be given as in section 4, equation (4.1), and suppose the conditions of lemma 3.2 above. Then we know by lemma 3.1 that the unit timelike future directed vector field

$$N^j = \frac{g^{0j}}{\sqrt{g^{00}}}$$

is everywhere orthogonal to the leaf $\mathcal{L}_t = \widetilde{\varphi}^{-1}(\{ct\} \times U_t) \subset \mathcal{M}$. Then $J_i = T_{ij} N^j$ is the energy-momentum vector density, and $T^{00}/g^{00} = T_{ij} N^i N^j$ is the local energy density. The total energy $E = E(t)$ with respect to the hypersurface U_t is given by

$$E = \int_{\tilde{\varphi}^{-1}(U_t)} \star \boldsymbol{J} = \int_{U_t} \frac{1}{g^{00}}\, T^{00} \mathrm{d}^{(3)}\boldsymbol{x}.$$

Here we are interested in the energy of a "test field" ϕ, i.e. a wave which responds but does not influence the background geometry of the space-time. Hence the total energy-momentum tensor T^{ij} is the sum of the unperturbed geometry part $T^{ij}_{(0)}$, whatever, and the wave part T^{ij}_ϕ, i.e. $T^{ij} = T^{ij}_{(0)} + T^{ij}_\phi$. The total energy of the wave field ϕ is then given by $E = E_\phi = \int_{U_t} T^{00}_\phi / g^{00} \mathrm{d}^{(3)}\boldsymbol{x}$.

This point of view can be compared to a possible interpretation of the Fermat principle in the context of light rays in space-times: In a gravitational field light propagates as if being in a medium of a certain index of refraction such that the rays are curved lines. Nonetheless, physical interpretations are not immediate and simple. E.g., the "speed of light in the medium" is not at all the physical speed measured by a local observer, cf. Stephani (1991), p. 108.

In the special case of an asymptotically flat space-time we have $N^j \to \partial_t$ as $r \to \infty$, and the energy density is asymptotically the one measured by a local observer γ with $\dot{\gamma} = \partial_t$. Consider a test field ϕ at time t with $\operatorname{supp}\phi$ in a small spacelike neighborhood of the asymptotic observer ("wave packet"). Then there is no difference between the total energy measured by the observer and the total energy with respect to U_t.

6. Massless spin wave operators

In this section we will see that massless fields with spin $s = \frac{n}{2}$, $n \in \{1, 2\}$, will be described by an operator

(6.1) $$L_{(s)} \colon C^\infty_0(U_t, \mathbb{C}^{2s+1}) \to C^\infty_0(U_t, \mathbb{C}^{2s+1})$$

in a chart $(\mathcal{U}, \tilde{\varphi})$ with $g^{00}(x) > 0 \;\; \forall x \in \mathcal{U}$. It is well-known that for $s = \frac{1}{2}$ and $s = 1$ the spin wave equation have a well posed initial value formulation, e.g. WALD (1984), pp. 375.

6.1. The Weyl operator

Massless spin-$\frac{1}{2}$ fields in a curved space-time are described by the *Weyl equation*

$$\begin{aligned} (l^j\partial_j + \varepsilon - \varrho)P^0 + (\bar{m}^j\partial_j + \pi - \alpha)P^1 &= 0, \\ (m^j\partial_j + \beta - \tau)P^0 + (n^j\partial_j + \mu - \gamma)P^1 &= 0, \end{aligned}$$

cf. CHANDRASEKHAR (1983), p. 544, eq. (105, 106). With the five complex matrices

(6.2) $$M_{(\frac{1}{2})}{}^j = \begin{pmatrix} l^j & \bar{m}^j \\ m^j & n^j \end{pmatrix} \quad \text{and} \quad Z_{(\frac{1}{2})} = \begin{pmatrix} \varepsilon - \varrho & \pi - \alpha \\ \beta - \tau & \mu - \gamma \end{pmatrix}$$

$(j = 0, 1, 2, 3)$ we may write instead

$$\big(M_{(\frac{1}{2})}{}^j\partial_j + Z_{(\frac{1}{2})}\big)\begin{pmatrix} P^0 \\ P^1 \end{pmatrix} = 0,$$

or equivalently

(6.3) $$M_{(\frac{1}{2})}{}^0\partial_0\phi = -\big(M_{(\frac{1}{2})}{}^\nu\partial_\nu + Z_{(\frac{1}{2})}\big)\phi$$

Especially for the Hermitian matrix $M_{(\frac{1}{2})}{}^0$ we can calculate by (2.1)

(6.4) $$\det\big(M_{(\frac{1}{2})}{}^0\big) = l^0n^0 - m^0\bar{m}^0 = \tfrac{1}{2}\,g^{00}.$$

Now let $(\mathcal{U}, \tilde{\varphi})$ be a chart satisfying (1.1). Then we have $M_{(\frac{1}{2})}{}^0 \in GL(2, \mathbb{C})$, i.e. $M_{(\frac{1}{2})}{}^0$ is invertible. Defining the *Weyl operator* $L_{(\frac{1}{2})} \colon C_0^\infty(U_t, \mathbb{C}^2) \to C_0^\infty(U_t, \mathbb{C}^2)$,

(6.5) $$L_{(\frac{1}{2})} := -c\big(M_{(\frac{1}{2})}{}^0\big)^{-1}\big(M_{(\frac{1}{2})}{}^\nu\partial_\nu + Z_{(\frac{1}{2})}\big)$$

as well as the *spinor functions* $\phi \in C_0^\infty(U_t, \mathbb{C}^2)$, $\phi := \begin{pmatrix} P^0 \\ P^1 \end{pmatrix}$, we may rewrite the Weyl equation (6.3) in the form

(6.6) $$\partial_t\phi = L_{(\frac{1}{2})}\phi.$$

The current spinor of the Weyl field is given, in Penrose notation, by $J^{AA'} = P^A\bar{P}^{A'}$ $(A, A' = 0, 1)$, cf. GÜVEN (1977), eq. (2.8). By the transformation law $J^j = \sigma_{AA'}{}^jJ^{AA'}$ with the Infeld-van der Waerden symbols written as matrices, $\sigma_{AA'}{}^j = \overline{M}_{(\frac{1}{2})}{}^j$ (CHANDRASEKHAR 1983, pp. 539, or PENROSE & RINDLER 1984, pp. 122), we see $J^j = l^jP^0\bar{P}^{0'} + m^jP^0\bar{P}^{1'} + \bar{m}^jP^1\bar{P}^{0'} + n^jP^1\bar{P}^{1'}$. Hence the energy density in U_t is J^0, i.e. the total energy E with respect to U_t is $E = \langle\phi, \phi\rangle$ with $Q = \frac{1}{g^{00}}\,\overline{M}_{(\frac{1}{2})}{}^0$, cf. Unruh (1974), eq. (2.9). By the relation $(l^0 - n^0)^2 + 4m^0\bar{m}^0 = (l^0 + n^0)^2 - 2g^{00}$ we see immediately that the eigenvalues of ${M_{(\frac{1}{2})}}^0$, hence the ones of $g^{00}Q$, are

$$\lambda_\pm = \frac{1}{2}\left(l^0 + n^0 \pm \sqrt{(l^0+n^0)^2 - 2\,g^{00}}\,\right). \tag{6.7}$$

We notice that they are positive, i.e. $\lambda_+ \geq \lambda_- > 0$, if and only if (1.1) is valid. Because now the eigenvalues of $\left({M_{(\frac{1}{2})}}^0\right)^{-1}$ are exactly $1/\lambda_\pm$ we may conclude for the maximal one

$$\lambda_{\max} = \frac{2}{l^0 + n^0 - \sqrt{(l^0+n^0)^2 - 2\,g^{00}}}\,. \tag{6.8}$$

By lemma 2.1 and equation (6.2) we have $\nabla_j {M_{(\frac{1}{2})}}^j = Z_{(\frac{1}{2})} + {Z_{(\frac{1}{2})}}^\star$, and thus $\nabla_j {M_{(\frac{1}{2})}}^j = \frac{1}{\sqrt{|g|}}\partial_j(\sqrt{|g|}{M_{(\frac{1}{2})}}^j)$. With $B_{(\frac{1}{2})} = -{M_{(\frac{1}{2})}}^\nu\partial_\nu$ and by (4.8) in theorem 4.5 we have $2\,\mathrm{Re}\,\langle\phi, (B_{(\frac{1}{2})} - Z_{(\frac{1}{2})})\phi\rangle = \langle\phi, H_{(\frac{1}{2})}\phi\rangle$, where

$$H_{(\frac{1}{2})} = \frac{1}{\sqrt{|g|}}\,\partial_\nu\big(\sqrt{|g|}\,{M_{(\frac{1}{2})}}^\nu\big) - Z_{(\frac{1}{2})} - {Z_{(\frac{1}{2})}}^\star = -\frac{1}{\sqrt{|g|}}\,\partial_0\left(\sqrt{|g|}\begin{pmatrix} l^0 & \bar{m}^0 \\ m^0 & n^0\end{pmatrix}\right).$$

If we denote $h_a = -\dfrac{1}{\sqrt{|g|}}\,\partial_0\big(\sqrt{|g|}\,{e_{(a)}}^0\big)$, we may write

$$H_{(\frac{1}{2})} = \begin{pmatrix} h_1 & h_4 \\ h_3 & h_2\end{pmatrix}, \tag{6.9}$$

and the eigenvalues are $\mu_{1/2} = \frac{1}{2}(h_1 + h_2 \pm \sqrt{(h_1+h_2)^2 - 4(h_1h_2 - h_3h_4)}\,)$. So, using theorem 4.5 we can state our final result for the Weyl operator (6.5) in curved space-times,

$$c\,\langle\phi, \lambda_{\max}\,\mu^-_{\min}\phi\rangle \leq 2\ \mathrm{Re}\ \langle\phi, L_{(\frac{1}{2})}\phi\rangle \leq c\,\langle\phi, \lambda_{\max}\,\mu^+_{\max}\phi\rangle. \tag{6.10}$$

6.2. The Maxwell operator

Maxwell's equations in curved space-time are expressed in the Newman-Penrose formalism by

$$\begin{aligned} l^j\partial_j\phi_1 - \bar{m}^j\partial_j\phi_0 &= (\pi - 2\alpha)\phi_0 + 2\varrho\phi_1 - \kappa\phi_2, \\ l^j\partial_j\phi_2 - \bar{m}^j\partial_j\phi_1 &= -\lambda\phi_0 + 2\pi\phi_1 + (\varrho - 2\varepsilon)\phi_2, \\ m^j\partial_j\phi_1 - n^j\partial_j\phi_0 &= (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2, \\ m^j\partial_j\phi_2 - n^j\partial_j\phi_1 &= -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2, \end{aligned} \tag{6.11}$$

with the three complex Maxwell scalars ϕ_0, ϕ_1, ϕ_2, e.g. Chandrasekhar (1983), Kramer *et al.* (1980). Subtracting the fourth from the first equation and reversing the sign of the third one we receive the system

$$\begin{pmatrix} n^j & -m^j & 0 \\ -\bar{m}^j & l^j + n^j & -m^j \\ 0 & -\bar{m}^j & l^j\end{pmatrix}\partial_j\begin{pmatrix}\phi_0\\ \phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix} 2\gamma - \mu & -2\tau & \sigma \\ \pi - 2\alpha + \nu & 2(\varrho - \mu) & 2\beta - \tau - \kappa \\ -\lambda & 2\pi & \varrho - 2\varepsilon\end{pmatrix}\begin{pmatrix}\phi_0\\ \phi_1\\ \phi_2\end{pmatrix}.$$

With the four Hermitian matrices

$$(6.12)\qquad {M_{(1)}}^j = \begin{pmatrix} n^j & -\frac{1}{\sqrt{2}}\, m^j & 0 \\ -\frac{1}{\sqrt{2}}\, \bar{m}^j & \frac{1}{2}\,(l^j+n^j) & -\frac{1}{\sqrt{2}}\, m^j \\ 0 & -\frac{1}{\sqrt{2}}\, \bar{m}^j & l^j \end{pmatrix},$$

$j = 0,1,2,3$, and with the notations

$$(6.13)\quad Z_{(1)} = \begin{pmatrix} \mu-2\gamma & \sqrt{2}\,\tau & -\sigma \\ \frac{1}{\sqrt{2}}\,(2\alpha-\pi-\nu) & \mu-\varrho & \frac{1}{\sqrt{2}}\,(\tau-2\beta+\kappa) \\ \lambda & -\sqrt{2}\,\pi & 2\varepsilon-\varrho \end{pmatrix},\ \phi = \begin{pmatrix} \phi_0 \\ \sqrt{2}\phi_1 \\ \phi_2 \end{pmatrix}$$

this means $({M_{(1)}}^j\partial_j + Z_{(1)})\phi = 0$, or

$$(6.14)\qquad {M_{(1)}}^0\partial_0\phi = -\big({M_{(1)}}^\nu\partial_\nu + Z_{(1)}\big)\phi,$$

cf. (6.3). By (6.12) we have

$$\det {M_{(1)}}^0 = \frac{1}{2}\,(l^0+n^0)(l^0n^0 - m^0\bar{m}^{\,0}) = \frac{1}{4}\,(l^0+n^0)\, g^{00}.$$

Thus in a coordinate chart with $g^{00} > 0$ we have ${M_{(1)}}^0 \in GL(3,\mathbb{C})$, and we can define the *Maxwell operator* $L_{(1)}\colon C_0^\infty(U_t,\mathbb{C}^3) \to C_0^\infty(U_t,\mathbb{C}^3)$,

$$(6.15)\qquad L_{(1)} := -c\big({M_{(1)}}^0\big)^{-1}\big({M_{(1)}}^\nu\,\partial_\nu + Z_{(1)}\big)$$

and the *spinor functions* $\phi \in C_0^\infty(U_t,\mathbb{C}^3)$, where ϕ is a column vector as in (6.13). Hence from Maxwell's equations we deduce the evolution equation

$$(6.16)\qquad \partial_t\phi = L_{(1)}\phi.$$

From $\det\big(\lambda I_3 - {M_{(1)}}^0\big) = \frac{1}{2}\big((\lambda-n^0)(\lambda-l^0) - m^0\bar{m}^{\,0}\big)\big(2\,\lambda - (l^0+n^0)\big)$ we calculate the eigenvalues of ${M_{(1)}}^0$,

$$(6.17)\qquad \lambda_1 = \tfrac{1}{2}\,(l^0+n^0), \qquad \lambda_\pm = \tfrac{1}{2}\big(l^0+n^0 \pm \sqrt{(l^0+n^0)^2 - 2\,g^{00}}\,\big).$$

They are positive on $\mathcal{U}$. The maximal eigenvalue of $\big({M_{(1)}}^0\big)^{-1}$ is

$$(6.18)\qquad \lambda_{\max} = \frac{2}{l^0+n^0-\sqrt{(l^0+n^0)^2-2g^{00}}}\,,$$

cf. (6.8). By (1.1) we have $(l^0+n^0)^2 - 2g^{00} < \big(l^0+n^0-\frac{g^{00}}{l^0+n^0)}\big)^2$, hence

$$(6.19)\qquad \lambda_{\max} < 2\,\frac{l^0+n^0}{g^{00}}\,.$$

According to CHANDRASEKHAR (1983), p. 422, (235), the energy-momentum tensor in the Newman-Penrose formalism is given by $T^{ij} = T^{ij}(\phi) = \phi^\star A^{ij}\phi$ with

$$A^{ij} = \frac{1}{4\pi}\begin{pmatrix} n^i n^j & -\sqrt{2}\, n^{(i}m^{j)} & m^i m^j \\ -\sqrt{2}\, n^{(i}\bar{m}^{j)} & l^{(i}n^{j)} + m^{(i}\bar{m}^{j)} & -\sqrt{2}\, l^{(i}m^{j)} \\ \bar{m}^i\bar{m}^j & -\sqrt{2}\, l^{(i}\bar{m}^{j)} & l^i l^j \end{pmatrix}.$$

For $Q := A^{00}/g^{00}$ the principal minors are $\frac{(n^0)^2}{4\pi g^{00}}$, $Q_{33} = \frac{(n^0)^2}{16\pi^2 g^{00}}$, $\det Q = \frac{1}{8\pi^3}$; i.e. Q is positive definite, if and only if (1.1) is valid. Thus the total energy of a Maxwell spinor $\phi \in C_0^\infty(U_t, \mathbb{C}^3)$ with respect to U_t is given by

$$E = E(\phi) = \int_{U_t} \phi^\star Q \phi \; \mathrm{d}^{(3)}\boldsymbol{x}.$$

We mention that both A_{00} and A_0^0 are *not* positive definite, if g_{00} gets non-positive, as can be shown easily with the respective principal minors. One might argue that it is rather unphysical to assume $g_{00} \le 0$, but this must be strongly rejected: the ergosphere of a Kerr-Newman black indeed has this property, as will be seen below. Moreover, this definition arises most naturally considering the Maxwell scalars as the dyad components of the 3-dimensional complex bispinors ϕ_{AB}, e.g. CARMELI (1977), p. 174: T^{ij} is exactly the electromagnetic energy-momentum spinor $T_{AA'BB'} = \frac{1}{2\pi}\,\phi_{AB}\bar{\phi}_{A'B'}$ (PENROSE & RINDLER 1984, pp. 325), rewritten with the aid of the Infeld-van der Waerden symbols, analogously to the Weyl case above. In flat Minkowski space-time the energy density becomes the usual one: Defining the two real 3-vector fields $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{B} = (B_x, B_y, B_z)$ by $\phi_0 = \frac{1}{\sqrt{2}}(B_y - E_x + \mathrm{i}(E_y + B_x))$, $\phi_1 = \frac{1}{\sqrt{2}}(E_z - \mathrm{i}B_z)$, $\phi_2 = \frac{1}{\sqrt{2}}(E_x + B_y + \mathrm{i}(E_y - B_x))$, (STEPHANI 1991, p. 167), we see with the Newman-Penrose tetrad of Minkowski space,

$$l^j = n_j = \frac{1}{\sqrt{2}}(1,0,0,1),\;\; n^j = l_j = \frac{1}{\sqrt{2}}(1,0,0,-1),\;\; m^j = -m_j = \frac{1}{\sqrt{2}}(0,1,i,0),$$

cf. DE VRIES (1994), that $Q = \frac{1}{8\pi} I_3$ and thus

$$\phi^\star Q \phi = \frac{1}{8\pi}\;(\bar{\phi}_0, \sqrt{2}\;\bar{\phi}_1, \bar{\phi}_2)\begin{pmatrix}\phi_0 \\ \sqrt{2}\;\phi_1 \\ \phi_2\end{pmatrix} = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2).$$

This is the usual energy density of the ($\boldsymbol{E}$,$\boldsymbol{B}$)-field, e.g. LANDAU and LIFSCHITZ (1975).

With lemma 2.1, $H_{(1)} = (h_{\alpha\beta})_{\alpha,\beta=1,2,3}$ in (20) is, analogously to $H_{(\frac{1}{2})}$ above, given by

$$
\begin{aligned}
h_{11} &= 2\,\mathrm{Re}\,\gamma - \frac{1}{\sqrt{|g|}}\,\partial_0(\sqrt{|g|}\,n^0),\\
h_{12} &= \frac{1}{\sqrt{2}}\Big(\bar\nu - \tau - \beta - \bar\alpha + \frac{1}{\sqrt{|g|}}\,\partial_0(\sqrt{|g|}\,m^0)\Big),\\
h_{13} &= \sigma - \bar\lambda,\\
h_{22} &= \mathrm{Re}(\varepsilon + \varrho - \mu - \gamma) - \frac{1}{2\sqrt{|g|}}\,\partial_0\big(\sqrt{|g|}\,(l^0 + n^0)\big),\\
h_{23} &= \frac{1}{\sqrt{2}}\,\big(\beta + \bar\alpha + \bar\pi - \kappa + \frac{1}{\sqrt{|g|}}\,\partial_0(m^0)\big),\\
h_{33} &= -2\,\mathrm{Re}\,\varepsilon - \frac{1}{\sqrt{|g|}}\,(\sqrt{|g|}\,l^0).
\end{aligned}
\tag{6.20}
$$

Thus $2\,\mathrm{Re}\langle\phi, L\phi\rangle = c\langle\phi, (M^0)^{-1}H_{(1)}\phi\rangle$ for a Maxwell spinor $\phi \in C_0^\infty(U_t, \mathbb{C}^3)$, and with $\mu_{\min}$ and $\mu_{\max}$ the minimal and maximal eigenvalues of $H_{(1)}$ and $\lambda_{\max}$ the maximal one of $(M^0)^{-1}$ as given by (6.18), we conclude

$$
c\,\langle\phi, \lambda_{\max}\,\mu_{\min}^{-}\phi\rangle \le 2\,\mathrm{Re}\langle\phi, L_{(1)}\phi\rangle \le c\,\langle\phi, \lambda_{\max}\,\mu_{\max}^{+}\phi\rangle. \tag{6.21}
$$

7. The Kerr-Newman space-time

Let $(\mathcal{M}, \boldsymbol{g})$, $\mathcal{M} \cong \mathbb{R} \times \mathbb{R}^3\backslash\{0\}$, be the Kerr-Newman space-time in Boyer-Lindquist coordinates $(ct, r, \theta, \varphi) \in \mathbb{R} \times (0,\infty) \times (0,\pi) \times (0, 2\pi)$. The contravariant components of the metric tensor are given in each point by the matrix

$$
g^{ij} = \begin{pmatrix}
\frac{\Sigma}{\rho\bar\rho\Delta} & 0 & 0 & \frac{(2Mr - Q^2)a}{\rho\bar\rho\Delta}\\
0 & -\frac{\Delta}{\rho\bar\rho} & 0 & 0\\
0 & 0 & -\frac{1}{\rho\bar\rho} & 0\\
\frac{(2Mr - Q^2)a}{\rho\bar\rho\Delta} & 0 & 0 & -\frac{\Delta - a^2\sin^2\theta}{\rho\bar\rho\Delta\sin^2\theta}
\end{pmatrix}
$$

with the functions

$$
\begin{gathered}
\Delta = \Delta(r) = r^2 - 2Mr + a^2 + Q^2, \qquad \rho = \rho(r,\theta) = r + \mathrm{i}a\cos\theta,\\
\Sigma = \Sigma(r,\theta) = \rho\bar\rho(r^2 + a^2) + (2Mr - Q^2)a^2\sin^2\theta
\end{gathered}
\tag{7.1}
$$

and the constants $M, a, Q \in \mathbb{R}$, $M \ge 0$, satisfying the condition

$$
a^2 + Q^2 \le M^2. \tag{7.2}
$$

M specifies the mass, a the rotation, and Q the charge of the Kerr-Newman space-time, describing a black hole as long as (7.2) holds. Consider the *Kinnersley tetrad*

$$
\begin{aligned}
&l^j = \frac{1}{\sqrt{2}\Delta}(r^2{+}a^2, \Delta, 0, a), \qquad l_j = \frac{1}{\sqrt{2}\Delta}(\Delta, -\rho\bar\rho, 0, -a\Delta\sin^2\theta),\\
&n^j = \frac{1}{\sqrt{2}\rho\bar\rho}(r^2{+}a^2, -\Delta, 0, a), \quad n_j = \frac{1}{\sqrt{2}\rho\bar\rho}(\Delta, \rho\bar\rho, 0, -a\Delta\sin^2\theta),\\
&m^j = \frac{1}{\sqrt{2}\rho}(\mathrm{i}a\sin\theta, 0, 1, \frac{\mathrm{i}}{\sin\theta}), \; m_j = \frac{1}{\sqrt{2}\rho}(\mathrm{i}a\sin\theta, 0, -\rho\bar\rho, -\mathrm{i}(r^2{+}a^2)\sin\theta),
\end{aligned}
\tag{7.3}
$$

cf. Kalnins & Williams (1990). In this tetrad the spin coefficients are given by

$$
\begin{gathered}
\varepsilon = \kappa = \sigma = \nu = \lambda = 0,\\
\beta = \frac{\cot\theta}{2\sqrt{2}\rho}, \qquad \varrho = -\frac{1}{\sqrt{2}\bar\rho}, \qquad \tau = -\frac{\mathrm{i}a\sin\theta}{\sqrt{2}\rho\bar\rho}, \qquad \pi = \frac{\mathrm{i}a\sin\theta}{\sqrt{2}\bar\rho^{\,2}},\\
\mu = -\frac{\Delta}{\sqrt{2}\rho\bar\rho^{\,2}}, \qquad \alpha = \pi - \bar\beta, \qquad \gamma = \mu + \frac{r-M}{\sqrt{2}\rho\bar\rho}.
\end{gathered}
\tag{7.4}
$$

If $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$ are the two zeroes of Δ, the hypersurface $r \equiv r_+$ is the *event horizon*, and $r \equiv r_e = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$ the *ergo horizon* of the black hole. Notice $r_e \ge r_+ \ge 0$ with (7.2). Further let

$$U = \{(ct, r, \theta, \varphi) \in \mathbb{R} \times (r_+, \infty) \times S^2\}.$$

Then the *outer space* of the black hole is the pre-image $\mathcal{U} \subset \mathcal{M}$ of U under the coordinate map $x \mapsto (ct, r, \theta, \varphi)$. We easily see $\rho\bar\rho > 0$, $\Delta > 0$ for $r > r_+$, thus

$$0 < \rho\bar\rho\Delta = \rho\bar\rho(r^2 + a^2) - (2Mr - Q^2)\rho\bar\rho \le \Sigma.$$

Hence we have, with $g^{00} = \Sigma/\rho\bar\rho\Delta$,

$$g^{00}(x) \ge 1 \qquad \forall x \in \mathcal{U}. \tag{7.5}$$

Thus $\mathcal{U}$ satisfies condition (1.1). $U_t = \{(r, \theta, \varphi) \mid (ct, r, \theta, \varphi) \in U\}$ with fixed $t \in \mathbb{R}$ is a connected open subset of $\mathbb{R}^3$, as in (4.1). The chart domain of U_t, the leaf $\mathcal{L}_t = \tilde\varphi^{-1}(U_t)$, is a partial Cauchy surface of $\mathcal{M}$, cf. Hawking & Ellis (1973).

Because of the stationarity of the Kerr-Newman space-time we have by (6.9) $H_{(\frac12)} = 0$, and thus we conclude for the Weyl operator $L_{(\frac12)}$ (6.5) by (6.10),

$$\operatorname{Re}\langle \phi, L_{(\frac12)}\phi\rangle = 0, \tag{7.6}$$

cf. de Vries (1994). More complicated is the calculation of the energy norm for the Maxwell operator. Checking up $(r^2 + a^2)(\rho\bar\rho + \Delta)/\Sigma \le 2(r^2 + a^2)\rho\bar\rho/\Sigma \le 2$, we find with (7.1) and (7.3) $(l^0 + n^0)/g^{00} \le \sqrt{2}$, i.e. by (6.19)

$$\lambda_{\max} \le 2\sqrt{2}. \tag{7.7}$$

Because of

$$\operatorname{Re}\varrho = -\frac{r}{\sqrt{2}\rho\bar{\rho}}, \quad \operatorname{Re}\mu = -\frac{r\Delta}{\sqrt{2}\rho^2\bar{\rho}^{\,2}}, \quad \operatorname{Re}\gamma = \frac{1}{\sqrt{2}\rho\bar{\rho}}\left(r - M - \frac{r\Delta}{\rho\bar{\rho}}\right),$$

we have by (6.20)

$$h_{22} = -\frac{2r(\rho\bar{\rho} - \Delta) - M\rho\bar{\rho}}{\sqrt{2}\rho^2\bar{\rho}^{\,2}}. \tag{7.8}$$

We see immediately $h_{13} = h_{33} = 0$. Because of $h_{12} = -(\bar{\pi} + \tau)/\sqrt{2}$ we have

$$h_{12} = \frac{\mathrm{i}a\sin\theta}{2\rho}\Big(\frac{1}{\bar{\rho}} + \frac{1}{\rho}\Big) = \frac{\mathrm{i}ar\sin\theta}{\rho^2\bar{\rho}},$$

and similarly $h_{23} = \sqrt{2}\,\bar{\pi}$, i.e.

$$h_{23} = -\frac{\mathrm{i}a\sin\theta}{\rho^2}.$$

With the ansatz $\phi_k = f_k(r,\theta)\,\mathrm{e}^{\mathrm{i}(\omega t + m\varphi)}$, $k = 0,1,2$, $\omega \in (0,\infty)$, $m \in \mathbb{Z}$, and noticing $n^j = \frac{1}{\sqrt{2}}(1,-1,0,0) + O(r^{-1})$-terms, $m^j = O(r^{-1})$, $2\gamma - \mu = O(r^{-1})$, $\tau = O(r^{-2})$ and $\sigma = 0$, we recognize the third one of Maxwell's equations (6.11) to be, up to $O(r^{-1})$-terms, $\partial_t\phi_0 \approx c\partial_r\phi_0$, i.e. $\partial_r f_0 \approx \mathrm{i}\omega/c f_0$. Thus we have $\phi_0 \to$ const $\mathrm{e}^{\mathrm{i}(\omega(t+r/c)+m\varphi)}$ as $r \to \infty$. Therefore ϕ_0 represents an incoming wave. So, considering scattering processes of electromagnetic waves measured in the asymptotic flat region $r \to \infty$ far away from the black hole, we may neglect the energy rate evolution of ϕ_0 in so far, as it does not interact with ϕ_1 and ϕ_2. Therefore we may set $h_{11} = 0$, hence by (6.20)

$$H_{(1)} = \begin{pmatrix} 0 & h_{12} & 0 \\ \bar{h}_{12} & h_{22} & h_{23} \\ 0 & \bar{h}_{23} & 0 \end{pmatrix}.$$

We achieve by $\det(\lambda I_3 - H_{(1)}) = \lambda\big(\lambda^2 - h_{22}\lambda - (|h_{12}|^2 + |h_{23}|^2)\big)$ the three eigenvalues $\lambda = 0$, $\lambda = \mu_+$, $\lambda = \mu_-$ where

$$\mu_\pm = \frac{1}{2}\,h_{22} \pm \sqrt{\Big(\frac{1}{2}\,h_{22}\Big)^2 + \frac{a^2\sin^2\theta}{\rho^2\bar{\rho}^{\,2}}\Big(1 + \frac{r^2}{\rho\bar{\rho}}\Big)}. \tag{7.9}$$

Now we try to estimate h_{22}. First we see $\rho\bar{\rho} - \Delta = 2\,Mr - a^2\sin^2\theta - Q^2$, i.e.

$$2r(\rho\bar{\rho} - \Delta) - M\rho\bar{\rho} = 3Mr^2 - 2Q^2r - a^2(2r\sin^2\theta + M\cos^2\theta).$$

With $r > M$ it follows $-2a^2r \le -a^2(2r\sin^2\theta + M\cos^2\theta) \le -a^2M$, therefore

$$3Mr^2 - 2Q^2r - 2a^2r \le 2r(\rho\bar{\rho} - \Delta) - M\rho\bar{\rho} \le 3Mr^2 - 2Q^2r - a^2M. \tag{7.10}$$

By (7.2) we have $rM(3r - 2M) = 3Mr^2 - 2M^2r \le 3Mr^2 - 2Q^2r - 2a^2r$, hence we get from (7.10) and (7.8)

$$-\frac{3Mr^2 - 2Q^2r - a^2M}{\sqrt{2}\rho^2\bar{\rho}^{\,2}} \;\le\; h_{22} \;\le\; -\frac{rM(3r - 2M)}{\sqrt{2}\rho^2\bar{\rho}^{\,2}} \le 0. \tag{7.11}$$

We notice that for $M > 0$ necessarily $h_{22} < 0$, and that $h_{22} = 0$, iff $M = 0$. We conclude from (7.9) $\mu_- \leq 0 \leq \mu_+$, and thus by (6.21) and (7.7)

$$2\sqrt{2}\, c\, \langle \phi, \mu_- \phi\rangle \leq 2\,\mathrm{Re}\langle \phi, L_{(1)}\phi\rangle \leq 2\sqrt{2}\, c\, \langle \phi, \mu_+ \phi\rangle. \tag{7.12}$$

Corollary 7.1. *Let $L_{(1)}$ be the Maxwell operator* (6.15) *in the Kerr-Newman space-time. Then we have the estimate*

$$-\frac{2c}{r_+^2}\Big(3M + 2|a|\Big)\,\|\phi\|^2 \leq 2\,\mathrm{Re}\langle \phi, L_{(1)}\phi\rangle \leq \frac{4c\,|a|}{r_+^2}\,\|\phi\|^2. \tag{7.13}$$

P r o o f. Because $\rho\bar\rho + r^2 = 2r^2 + a^2\cos^2\theta \leq 2r^2 + 2a^2\cos^2\theta = 2\rho\bar\rho$, and by the elementary inequality $p^2 + q^2 \leq (|p| + |q|)^2$ for any real numbers p, q, we may estimate the root term in (7.9) $\sqrt{\ldots} \leq \sqrt{2}\,|a|\sin\theta/\rho\bar\rho - h_{22}/2$ with (7.11). This means

$$0 \leq \mu_+ \leq \sqrt{2}\,\frac{|a|\sin\theta}{\rho\bar\rho}\,,$$

and $h_{22} - \sqrt{2}\,|a|\sin\theta/\rho\bar\rho \leq \mu_- \leq h_{22}$, or

$$-\frac{\sqrt{2}}{\rho\bar\rho}\left(\frac{3Mr^2 - 2Q^2 r - a^2 M}{2\rho\bar\rho} + |a|\sin\theta\right) \leq \mu_- \leq -\frac{rM(3r-2M)}{\sqrt{2}\rho^2\bar\rho^{\,2}}. \tag{7.14}$$

By $r_+^2 < r^2 \leq \rho\bar\rho$ and $(2Q^2 r + a^2 M)/\rho\bar\rho \to 0$ for $r \to \infty$ we have

$$-\frac{\sqrt{2}}{r_+^2}\left(\tfrac{3}{2}\,M + |a|\right) \leq \mu_- \leq \mu_+ \leq \sqrt{2}\,\frac{|a|}{r_+^2}.$$

(7.12) completes the proof. □

8. Conclusion

We notice that the upper bound of the energy change rate in (7.13) is positive in the rotating case, i.e. if $a \neq 0$. Thus an electromagnetic wave in the outer space of a Kerr-Newman black hole may gain energy. Hence our result provides an affirmation and even a quantitative control of the superradiance in the Kerr case $Q = 0$, first observed by Teukolsky (1973) and Starobinskii and Churilov (1973), after the pioneering works of Zel'dovich (1972) and Misner (1972). In the contrary, (7.6) shows that a Weyl neutrino field preserves energy, cf. Unruh (1974). However, in this context superradiance has been shown with the aid of the vector field ∂_t in Kerr space-time that is Killing everywhere, but gets spacelike in the ergosphere, whereas our considerations are based on the unit normal of the hypersurfaces $\{\mathrm{d}t = 0\}$ that is everywhere timelike and asymptotically Killing.

The methods developed here are also easily applicable to the context of the massive Dirac equation, as is done in (de Vries 1994). They might extend naturally to the case of massless spin-s waves with $s > 1$. So they may eventually open a door for analyzing interactions of curved space-times and spin, which lead in the context of a rotating black hole to a "filtering process" amplifying spin-n waves, and ignoring spin-$\frac{n}{2}$ waves ($n \in \mathbb{N}$), cf. CHANDRASEKHAR (1983).

Moreover the quantitative limits in eq. (7.6) for the rate of change of energy of electromagnetic fields in the outer space of a Kerr-Newman black hole may be used to estimate bounds for the mass of all black holes in the observable universe with aid of the cosmic background radiation anisotropy measurements of the *COBE* satellite, cf. (DE VRIES *et al.* 1995).

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Note added in proof: In de Vries et al., and in A. de Vries, The evolution of the Dirac field in curved space-times, *Manuscr. Math.* **88**, 233–246 (1995), there are references to equations given here: They are denoted as eq. (32), (33), (43), (44), (56) and (57) and refer to (6.9), (6.10), (6.20), (6.21), (7.12) and (7.13), respectively.

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