The arbitrage theorem in a discrete infinite-horizon market model

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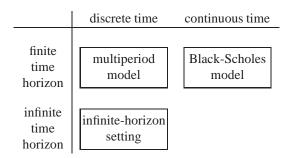
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1 Introduction

This paper is a short introduction to one of the three cornerstones of the theory of portfolio choice and asset pricing in multiperiod settings under uncertainty, which are *arbitrage*, *optimality of the agents' utilities*, and *market equilibrium*. These three notions form the basic constraints of asset pricing. The most important unifying principle is that any of these constraints implies certain "state prices," meaning positive discount factors, one for each time and each state. With them the price of any security turns out to be merely the state-price weighted sum of its future payoffs. This idea can be traced back to Kenneth Arrow's invention of the general equilibrium model of security markets in 1953.

So far, the theory covers three models which differ in their settings with respect to the time evolution, the finite multiperiod model, the dicrete infinite-horizon setting, and the Black-Scholes model. They are related according to the following graphic:

	discrete time	continuous time
finite time horizon	multiperiod model	Black-Scholes model
infinite time horizon	infinite-horizon setting	

The glamorous star among these theories surely is the Black-Scholes model. Introduced in 1973 with the option pricing formula by Fisher Black and Myrton Scholes, and proved in the same year by Robert Merton to hold on markets without arbitrage, it developed to a general derivative pricing theory. Roughly, it says that the security markets are in equilibrium without arbitrage, and the market prices evolve as continuous-time random processes superposed by a "white noise," a Brownian motion. As a great advantage, the mighty theoretical physics machinery called "Feynman-Kac formula" can be applied. For details see [3, 7, 8].

However, so far there does not exist an extension to an infinite horizon setting. Although the Black-Scholes model fits extraordinarily well to the pricing of derivatives (and also other securities such as bonds or credits) with definitive maturity dates, the potentially unlimited life of stocks may need the infinite horizon setting to be completely understood.

So in this paper we pay our attention to the discrete time models. There are several reasons not to underestimate them. For instance, in practice one measures market data at isolated points of time, such as daily quotations or weekly time series. In addition, numerical simulations such as Monte Carlo methods are inherently discrete.

Moreover, there are tendencies to apply game theory to the theory of markets, supposing each market participant as a player with individual strategies and utilities. In game theory [4], however, the moves made by the players only take place on discrete time dates.

The basic approach to the multiperiod model goes back to Kenneth Arrow in 1953, the final state-price implications were first mentioned by S. Ross in 1978. The results for the infinite-horizon setting are based on considerations of Darrel Duffie in his textbook [3].

2 What is arbitrage?

Arbitrage is "speculation" without risk.[1] In its simplest form in the theory of portfolio choice and asset pricing, it means taking simultaneous positions in different assets so that one is guaranteed a riskless profit higher than the riskless return, such as given by bonds like the US Treasury bills. If such profits exist, we say that there is an *arbitrage*, or an *arbitrage opportunity*.

Consider a stock that is traded on both the New York Stock Exchange and the London Stock Exchange. Suppose the stock price is $172 in New York and £100 in London at a time when the exchange rate is $1.75 per pound. An arbitrageur could simultanuously buy 100 shares in New York and sell them in London to obtain a risk-free profit of

$$100 \cdot (\pounds 100 \cdot 1.75\$/\pounds - \$172) = \$300$$

in the absence of transaction costs, cf. [6].

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[1]Etymologically, *arbitrage* derives from the French word for *regulation*, whose root is Latin *arbitrare* – to decide, to judge

To anticipate the notation below, the arbitrage portfolio θ in the market

$$S = \begin{pmatrix} S_{\mathrm{NY}} \\ S_{\mathrm{Lon}} \end{pmatrix}$$

is given by $\theta = (-n, n)$, with $n = 100$ say, and the corresponding payoff $\delta^{\theta} = \theta \cdot S = n(S_{\mathrm{Lon}} - S_{\mathrm{NY}}) = \$3n$. Hence $\delta^{\theta} > 0$, the portfolio yields "something for nothing."
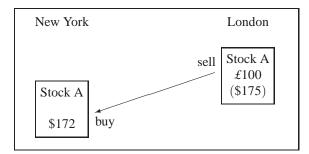

Figure 1: Arbitrage opportunity at exchange rate 1.75$/£.

In practice, of course, there are many arbitrage opportunities. This, however, does not reduce the general interest in arbitrage-free prices. In fact, they have enormous theoretical relevance, for the most important conclusions concerning the evolution of markets can only be made under the condition of non-arbitrage.

Arbitrage opportunities cannot last for long. As arbitrageurs in our example above buy the stock in New York, the forces of supply and demand will cause the dollar price of the stock to rise. Similarly, as they sell the stock in London, the sterling price of the stock will be driven down. Very quickly, the two prices will become equivalent at the current exchange rate.

The very existence of arbitrageurs means that in practice only very small arbitrage opportunities are observed in the prices that are quoted in financial markets. Arbitrageurs are "information catalysators," they make the market prices a reflected image of the available information.

2.1 The concept of fair prices

The notion of arbitrage is used to obtain a practical definition of a "fair price" or a "present value" for a financial asset. The price of a security is *fair*, or the security is *correctly priced*, if there are no arbitrage opportunities at those prices. Such arbitrage-free asset prices will be utilized as benchmarks, deviations from which indicate opportunities for excess profits.

Assume two discrete times $t = 0$ and $t = T$ at which trades can be made. Suppose moreover a financial market consisting solely of . . .

1. a risk-free bond β such as a Treasury Bill whose return until next period is $1 + r$;

2. an underlying asset, e.g., a stock S;

3. a call option on the underlying asset, with premium C and a strike price K, which expires next period. (The call option gives its holder the right, but not the obligation, to pay K for the stock, with divident, after the state is revealed.)

The market is represented by the vector

$$\mathbf{S}(t) = \begin{pmatrix} \beta(t) \\ S(t) \\ C(t) \end{pmatrix}.$$

To keep things simple we assume a world (at least that part of it which influences the financial market) in which there are only two state ω_- and ω_+ possible. They occur with probability p_- and p_+, respectively. This means that $\mathbf{p} = (p_-, p_+)$ is a probability vector where $p_- + p_+ = 1$. The market prices $E_0[\mathbf{S}(T)]$ *expected* at time $t = 0$ ("now") for the future $t = T$, denoted by, then are given as

$$E_0[\mathbf{S}(T)] = \begin{pmatrix} (1+r)\beta(0) & (1+r)\beta(0) \\ S_- & S_+ \\ (S_- - K)^+ & (S_+ - K)^+ \end{pmatrix} \cdot \begin{pmatrix} p_- \\ p_+ \end{pmatrix}. \quad (1)$$

Here $S_{\pm} = S(T, \omega_{\pm})$, and $x^+ := \max(x, 0)$ for any real number x. Notice that the payoff of the bond is the same in each state of the world, because it is riskless. The stock price $S(T)$, however, may assume the values S_- or S_+, depending on the states of the world $\omega_{\pm}$.

Defining $\psi_0 = (1 + r)$ and the vector $\vec{\psi} = \psi_0 \mathbf{p}$, we see that

$$\beta(0) = \frac{1}{\psi_0} E_0[\beta(T)]. \quad (2)$$

Thus $\beta(0)$ denotes the present value of the bond and ψ_0 is the discounting factor for the period $[0, T]$. It is called the "state price deflator" at time $t = 0$. Furthermore, $\psi_{\pm} = \psi_0 p_{\pm}$ are the "state prices" for each possible event. The arbitrage theorem below will tell us that the financial market is arbitrage-free if and only if

$$\mathbf{S}(0) = \frac{1}{\psi_0} E_0[\mathbf{S}(T)]. \quad (3)$$

If the security S pays 1 currency unit (cu) in state ω_-, and 0 cu in state ω_+, then $S_- = \psi_-$: The investors are willing to pay ψ_- for an "insurance policy" that offers 1 cu in state ω_- and nothing in state ω_+. Similarly, ψ_+ indicates how much investors would like to pay for an "insurance policy" that pays 1 cu in state ω_+ and nothing in state ω_-. Clearly, by spending $\psi_0 = \psi_- + \psi_2$ one can guarantee 1 unit of account in the future, regardless of which state is realized. This explains the interpretation of ψ_i as state prices.

There are the following important utilizations of arbitrage-free prices:

1. as benchmarks for determining market prices for new derivative products;

2. as benchmark prices for risk management "worst case scenario" simulations;

3. to "mark to market" the assets held in portfolios, i.e. to calculate the current market value of a non-liquid asset for which no trades have been observed lately;

4. as benchmark prices to be compared with observed actual trading prices; significant differences between observed and arbitrage-free values might indicate excess profit opportunities.

3 Probability theoretic setting

We investigate arbitrage and state prices in an abstract discrete infinite-horizon setting. For details I refer to the introduction [3], for an overview to the notions of probability theory see [1, 2, 8].

Suppose Ω to be the set of the *states of the world*, $\mathscr{F}$ a σ-algebra of Ω, and P a probability measure $P\colon \mathscr{F} \to [0,1]$, such that $(\Omega, \mathscr{F}, P)$ is a probability space.

Time will be represented by the variable t. We will suppose a discrete time model where t only obtains the (at most countably many) values $t_0, t_1, \ldots, t_j, t_{j+1}, \ldots$. We define $\mathscr{T} = \{t_0, t_1, \ldots\}$ as the time set.

For each date $t \in \mathscr{T}$, we construct a subalgebra $\mathscr{F}_t \subset \mathscr{F}$ of $\mathscr{F}$, such that the *filtration* $\mathbb{F} := \{\mathscr{F}_{t_0}, \mathscr{F}_{t_1}, \mathscr{F}_{t_2}, \ldots\}$ is a *filtration*. Hence at each time date t, $\mathscr{F}_t$ denotes the set of *measureable ("observable") states* at time t. At t a measureable state is *known* to be true or false. Thus $\mathscr{F}_t$ assigns the *information available at time t*. Being a filtration, $\mathscr{F}_s \subset \mathscr{F}_t$ whenever $s \leqq t$. That means an infinite memory, such that *past* states are never forgotten. At the starting time $t = t_0$ there is no information, so $P(A) = 0$ or 1 for every macrostate A in $\mathscr{F}_{t_0}$. In this way the filtration $\mathbb{F}$ represents the way how information is revealed through time.

The measurable states in $\mathscr{F}_t$ represent the questions that can be asked at time t. In a filtration the number of askable questions permanently increases. The potential *knowledge* gets finer and finer as time passes by.

Let L be the space of possible prices for each state of the world, given by the Hilbert space[2] $L^2(\Omega, \mathscr{F}, P)$ of real-valued square-integrable functions on Ω,

$$L = L^2(\Omega, \mathscr{F}, P) \qquad (4)$$

where $L^2(\Omega, \mathscr{F}, P) = \{f\colon \Omega \to \mathbb{R} : \int_\Omega |f(\omega)|^2 \, \mathrm{d}P(\omega) < \infty\}$. L then is a separable real Hilbert space. It has the inner product $\langle \cdot | \cdot \rangle\colon L \times L \to \mathbb{R}$,

$$\langle f | g \rangle = \int_\Omega f(\omega) g(\omega) \, \mathrm{d}P(\omega) = E[fg],$$

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[2]For a definition a Hilbert space, see [10]

where $E[X]$ denotes the expectation of a random variable $X \in L$.

If, e.g., the world can achieve only k discrete states, $\Omega = \{\omega_1, \ldots, \omega_k\}$, then $L \cong \mathbb{R}^k$. We then call Ω also a k-fold *alternative*.

An n-dimensional *stochastic process* X is a time-dependent random variable in $\mathbb{R}^n$. More accurately, a process X with time set $\mathscr{T}$ and probability space $(\Omega, \mathscr{F}, P)$ is a family $X = X(t, \cdot)_{t \in \mathscr{T}} \in L$ of random variables

$$X(t, \cdot)\colon \Omega \to \mathbb{R}^n \qquad (t \in \mathscr{T}). \qquad (5)$$

The image set is called configuration space. For details see [5] §2. An n-dimensional *adapted process* (with respect to $\mathbb{F}$) is a family $X = \{X(t_0), X(t_1), X(t_2), \ldots\}$ such that, for each t, $X(t)$ is an n-dimensional $\mathscr{F}_t$-measurable random variable with respect to $(\Omega, \mathscr{F}_t, P)$. Informally this means: At time t the state ω and thus the vector $X(t, \omega)$ is known. We denote the set of n-dimensional adapted processes by $\mathscr{V}_n$.

For $s, t \in \mathscr{T}$ and an adapted process $X\colon \mathscr{T} \times \Omega \to \mathbb{R}^n$ we let

$$E_t[X(s)] := E[X(s)|\mathscr{F}_t] \qquad (6)$$

denote the conditional expectation of $X(s)$ given the information $\mathscr{F}_t$. We note that $E_t[X(s)] = X(s)$ (almost surely) if $s \leqq t$. (This is a consequence of the fact that $X(s)$ is $\mathscr{F}_t$-measurable for any $s \leqq t$, see [2] §15, eq. (15.7).) If X moreover is a *martingale*, then by definition $E_t[X(s)] = X(t)$ if $s > t$. For a martingale we thus have shortly

$$E_t[X(s)] = \begin{cases} X(s) & \text{if } s \leqq t, \\ X(t) & \text{if } s > t. \end{cases} \qquad (7)$$

In the sequel we will restrict ourselves to the set $\mathscr{L}_n$ of mean-summable n-dimensional adapted processes given by

$$\mathscr{L}_n := \Big\{ X \in \mathscr{V}_n : E\Big[\sum_{t \in \mathscr{T}} X^2(t)\Big] < \infty \Big\}. \qquad (8)$$

We will denote especially $\mathscr{L} = \mathscr{L}_1$. By construction, at any time $t \in \mathscr{T}$ a process $X \in \mathscr{L}_n$ is in L^n. Because $E[\sum_t |X(t)|] < \infty$, by Fubini's theorem we can reverse the order of the expectation and the time integral, $E[\sum_t X(t)] = \sum_t E[X(t)]$, cf. [3] §C. Hence, $\mathscr{L}_n \subset L^2(\mathscr{T} \times \Omega, \mathbb{R}^n)$.

With the inner product $(\cdot | \cdot)_n\colon \mathscr{L}_n \times \mathscr{L}_n \to \mathbb{R}$ given by

$$(X | Y)_n := E\Big[\sum_{t \in \mathscr{T}} X(t) \cdot Y(t)\Big], \qquad (9)$$

we see that $\mathscr{L}_n$ is a Hilbert space. It is isomorphic to the classical sequence space l_2, $\mathscr{L}_n \cong l_2$, [10] §2.

What about the inner product (9)? We note so far that the square of X^2 of a process is determined by all its future expectations. What else are prices?

4 Arbitrage and state prices

Financial markets. We define a *financial market* consisting of n assets — such as options, futures, forwards, stocks, or bonds — to be the pair (δ, S) of two n-dimensional adapted processes $S, \delta \in \mathscr{L}_n$. Here $S(t) = (S_1(t), \ldots, S_n(t))$, where for $i = 1, \ldots, n$ each S_i denotes the *i-th security price process*, so that $S_i(t, \omega)$ is the price of the i-th security *ex dividend* at time t and in state ω. Additionally, security number i is a claim to a *divident* $\delta_i(t, \omega)$ denoting the divident paid by the security at time t in state ω. (That is, at each time t the security pays its divident $\delta_i(t)$ and is then available for trade at price $S_i(t)$.) We assume that the *divident process*, $\delta = (\delta_1, \ldots, \delta_n)$ is adapted, $\delta \in \mathscr{L}_n$. Hence for each time $t \in \mathscr{T}$ and state of the world $\omega \in \Omega$ the vectors of asset prices $S(t, \omega)$ and of dividends δ are given by

$$S(t, \omega) = \begin{pmatrix} S_1(t, \omega) \\ \vdots \\ S_n(t, \omega) \end{pmatrix}, \quad \delta(t, \omega) = \begin{pmatrix} \delta_1(t, \omega) \\ \vdots \\ \delta_n(t, \omega) \end{pmatrix}. \quad (10)$$

The *cum-divident* security price is detemined by $S(t_j) + \delta(t_j)$. The *added market price process* at time t_j is

$$D(t_j) = S(t_j) + \delta(t_j) - S(t_{j-1}) \quad (11)$$

where $S(t_{-1}) := 0$, as well as $\delta(t_0) = 0$. In other words we have $D(t_0) = S(t_0)$. In [3] the *gain process* $G(t)$ is defined as

$$G(t_j) = S(t_j) + \sum_{l=0}^{j} \delta(t_l) = \sum_{l=0}^{j} D(t_l). \quad (12)$$

Trading strategies. A *trading strategy* is an n-dimensional adapted process $\theta \in \mathscr{L}_n$ such that for any financial market S the product $\theta \cdot S$ is an adapted process, $\theta \cdot S \in \mathscr{L}$. Here $\theta(t, \omega) = (\theta_1(t, \omega), \ldots, \theta_n(t, \omega)) \in \mathbb{R}^n$ represents the *portfolio* held after trading at time t and in state ω. Let $\Theta \subset \mathscr{L}_n$ denote the given set of possible trading strategies. The *market value process* V^θ of the trading strategy is the adapted process

$$V^\theta(t) = \theta(t) \cdot S(t). \quad (13)$$

By construction $V^\theta \in \mathscr{L}$. The *payoff process* δ^θ generated by the trading strategy θ, for $t_j \geq t_0$, is defined by

$$\boxed{\delta^\theta(t_j) = \theta(t_{j-1}) \cdot [S(t_j) + \delta(t_j)] - V^\theta(t_j),} \quad (14)$$

with "$\theta(t_{-1})$" taken to be zero by convention. By construction, for $\theta \in \Theta$ we have that δ^θ is a one-dimensional adapted process $\delta^\theta \in \mathscr{L}$. Especially, for each $t \in \mathscr{T}$ we have $\delta^\theta(t) \in L$. For future purposes we note that

$$\begin{aligned} \sum_{l=j}^{\infty} \delta^\theta(t_l) &= \sum_{l=j}^{\infty} [\theta(t_{l-1}) - \theta(t_l)] S(t_l) \\ &\quad + \sum_{l=j}^{\infty} \theta(t_{l-1}) \delta(t_l) \\ &= \theta(t_{j-1}) S(t_j) + \sum_{l=j}^{\infty} \theta(t_l)[S(t_l) - S(t_{l-1})] \\ &\quad + \sum_{l=j}^{\infty} \theta(t_{l-1}) \delta(t_l) \\ &= \theta(t_{j-1}) S(t_j) + \sum_{l=j}^{\infty} \theta(t_l) D(t_l). \end{aligned}$$

Especially for $j = 0$ we have with $\theta(t_{-1}) = 0$ that $\sum_{l=0}^{\infty} \delta^\theta(t_l) = \sum_{l=0}^{\infty} \theta(t_l) D(t_l)$, i.e.

$$E_{t_j}\Big[\sum_{l=j+1}^{\infty} \delta^\theta(t_l) \Big] = E_{t_j}\Big[\sum_{l=j+1}^{\infty} \theta(t_l) D(t_l) \Big]. \quad (15)$$

Example 4.1 Let be $s, T \in \mathscr{T}$. Consider the simple trading strategy of buying asset number i at time $t = s$ and selling it at time T, i.e. $\theta_l(t) = 0$ for $l \neq i$ and for each $t \in \mathscr{T}$, as well as $\theta_i(t) = 1$ for $t < T$ and $\theta_i(t) = 0$ for $t \leq s$ and for $t \geq T$. Then the payoff process δ^θ generated by the trading strategy is given simply by

$$\delta^\theta(t) = \begin{cases} -S_i(s) & \text{if } t = s, \\ \delta_i(t) & \text{if } s < t < T \\ S_i(T) & \text{if } t = T \\ 0 & \text{else.} \end{cases} \quad (16)$$

State-price deflators. Let $\mathscr{L}_+ = \{\psi \in \mathscr{L}: \psi(t, \omega) \geq 0 \ \forall(t, \omega)\}$ denote the cone of non-negative adapted processes in $\mathscr{L}$.[3] The "interior" $\mathscr{L}_+^\circ$ of the cone,

$$\mathscr{L}_+^\circ := \{\psi \in \mathscr{L} : \psi(t, \omega) > 0 \quad \forall(t, \omega)\} \quad (17)$$

is called the set of *deflators*. ("$\mathscr{L}_+ \setminus \partial\mathscr{L}_+$" – but what topology?) Thus a deflator is a strictly positive one-dimensional adapted process. A deflator ψ is a *state-price deflator* for the divident-price pair (δ, S), if for all $t_j \in \mathscr{T}$ and every state ω of the world,

$$\boxed{S(t_j, \omega) = \frac{1}{\psi(t_j, \omega)} E_{t_j}\Big[\sum_{l=j+1}^{\infty} \psi(t_l) \cdot D(t_l) \Big].} \quad (18)$$

Note that this is a vector-valued equation. With (15) we see that a deflator ψ is a state-price deflator if and only if for any trading strategy θ

$$V^\theta(t_j, \omega) = \frac{1}{\psi(t_j, \omega)} E_{t_j}\Big[\sum_{l=j+1}^{\infty} \psi(t_l) \cdot \delta^\theta(t_l) \Big]. \quad (19)$$

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[3] A *cone* is a subset C of a linear space with the property that for every $\psi \in C$ and every positive constant λ we have also $\lambda\psi \in C$.

This means roughly that the market value $V^\theta = \theta \cdot S$ of a trading strategy is, at any time, the state-price discounted expected future payoffs generated by the strategy.

For a finite horizon setting $\mathscr{L}_n^T := \{X \in \mathscr{L}_n : X(t) = 0$ for $t > T\}$, the right-hand side of (19) vanishes, and so $S(T) = 0$.

Arbitrage. For any strategies θ, $\theta' \in \Theta$ and scalars a, $b \in \mathbb{R}$, we have $a\delta^\theta + b\delta^{\theta'} = \delta^{a\theta + b\theta'}$. Thus the *marketed subspace* $\mathscr{M} = \mathrm{span}(\delta^\theta : \theta \in \Theta)$ of divident processes generated by trading strategies Θ is a linear subspace of the Hilbert space $\mathscr{L}$ of mean-summable adapted processes, $\mathscr{M} \subset \mathscr{L}$.

Given a divident-price pair (δ, S) for n securities, a trading strategy θ is an *arbitrage* if $\delta^\theta > 0$. This means $P\big(\delta^\theta(t) > 0\big) > 0$ for at least one time t, and $\delta^\theta(t) \geqq 0$ for all $t \in \mathscr{T}$.

Geometrically the arbitrage condition means the following: Both the cone of non-negative processes $\mathscr{L}_+$ and the marketed subspace $\mathscr{M}$ are closed convex subsets of $\mathscr{L}$. Hence there is no arbitrage if and only if $\mathscr{L}_+ \cap \mathscr{M} =$
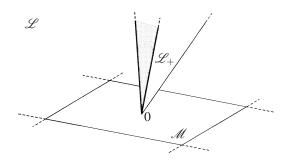

Figure 2: The marketed subspace $\mathscr{M}$ and the deflator cone $\mathscr{L}_+$.

$\{0\}$, see figure 2. In this way the deflator cone $\mathscr{L}_+$ may also be called the "cone of arbitrage."

Arbitrage Theorem. *The financial market (δ, S) admits no arbitrage if and only if there is a state-price deflator for (δ, S).*

Proof. With the Separating Hyperplane Theorem [9], for two closed convex subsets $\mathscr{L}_+$ and $\mathscr{M}$ with $\mathscr{L}_+ \cap \mathscr{M} = \{0\}$ there exists a linear functional $f \colon \mathscr{L} \to \mathbb{R}$ with $f(Y) < f(X)$ for all $Y \in \mathscr{M}$ and $X \in \mathscr{L}_+$, $X \neq 0$. Since $\mathscr{M}$ is a linear space, this implies $f(\mathscr{M}) = 0$, and $f(X) > 0$ for all nonzero $X \in \mathscr{L}_+$.

According to the Riesz representation theorem [9] §II.2, for each linear functional $f \colon \mathscr{L} \to \mathbb{R}$ there exists a unique vector $\psi \in \mathscr{L}$, called the Riesz representation of f, such that $f(X) = (\psi|X)$ for all $X \in \mathscr{L}$, i.e.

$$f(X) = E\Big[\sum_{l=0}^{\infty} \psi(t_l)X(t_l)\Big], \quad \forall X \in \mathscr{L}. \qquad (20)$$

Since f is positive, ψ is a deflator.

First, if a state-price deflator exists, equation (19) yields $f(\delta^\theta) = \psi(t_{-1})V^\theta(t_{-1}) = 0$. So let us consider the "only if"-direction. We have $E[\sum_t \psi(t)\delta^\theta(t)] = 0$ for any trading strategy θ, for $\delta^\theta \in \mathscr{M}$. Especially for the i-th security with $i \in \{1, \ldots, n\}$ the trading strategy in example 4.1 we have with (16)

$$E\Big[\psi(\tau)S_i(\tau) + \sum_{t_0 < t < \tau} \psi(t)\delta_i(t) - \psi(t_0)S_i(t_0)\Big] = 0.$$

The "deflated gain process" of the i-th security $G_i^\psi(t) = \psi(t)S_i(t) + \sum_{s > t_0}^{t} \psi(s)\delta_i(s)$ thus is a martingale, since $E[G_i^\psi(\tau)] = \psi(t_0)S_i(t_0)$, and τ is arbitrary. This implies that $E_t[G_i^\psi(s)] = \psi(t)S_i(t)$ for any $s > t$. Because this is valid for any $i = 1, \ldots, n$, equation (19) is satisfied, and ψ is a state-price deflator. $\qquad \square$

5 Discussion

The arbitrage theorem tells us two things:

(i) Evidently, in case of non-arbitrage the *expectation* of future prices determines the *actual* market prices. In other words, actual prices express future expectations.

(ii) The actual market prices in turn determine the future market expectations. (However, they do not yield the precise probability distributions.)

In the discrete finite time setting, $\mathscr{T} = [0, T]$, (the "basic multiperiod model") it can be shown that there exists no arbitrage if and only if there is an "equivalent" martingale measure. Two probability measures p and q are called equivalent if p and q assign zero probabilities to the same states or events: $p(x) = 0 \iff q(x), x \in \mathscr{F}$, see [3] §2G.

This version of the arbitrage theorem builds the bridge to the continuous time setting, the "Black-Scholes market", where Brownian motion influences the market prices as a white noise. With some mild and technical restrictions there is no arbitrage in a Black-Scholes market, if and only if there exist an equivalent martingale measure.

Open questions that remain:

(i) How can the phenomenon of emerging and vanishing securities over an infinite time horizon can be tackled?

(ii) Is there a similar approach with a continuous time but infinite horizon?

(iii) What about discrete prices? For why should security prices supposed to be continuous? Perhaps a permanent discrete "jump" assumption is much more realistic and may lead to a deeper understanding of the nature of markets. In fact, real prices only achieve certain discrete base points or "ticks:" They *are not* continuous. But how does the corresponding configuration space look like? Our setting above would not yield a Hilbert space L …

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